

# The exponential of the spin representation of the Lorentz algebra

jason hanson\*

January 31, 2012

## Abstract

As discussed in a previous article, any (real) Lorentz algebra element possess a unique orthogonal decomposition as a sum of two mutually annihilating decomposable Lorentz algebra elements. In this article, this concept is extended to the spin representation of the Lorentz algebra. As an application, a formula for the exponential of the spin representation is obtained, as well as a formula for the spin representation of a proper orthochronous Lorentz transformation.

## 1 Orthogonal decomposition

Let  $g$  be a Lorentz metric on  $\mathbb{R}^4$ . That is,  $g$  is a symmetric nondegenerate inner product with determinant  $-1$ . For our purposes, we need not specify a signature for  $g$ . The *Lorentz group*  $O(g)$  is the Lie group of linear transformations  $\Lambda$  on  $\mathbb{R}^4$  such that  $\Lambda^T g \Lambda = g$ , and the *Lorentz algebra*  $so(g)$  is the Lie algebra of transformations  $L$  for which  $L^T g + g L = 0$ .

### 1.1 Decomposition of bivectors

Elements of  $so(g)$  are called *Lorentz bivectors*. A special type of Lorentz bivector is the **simple bivector**, which takes the form  $u \wedge^g v$  for four-vectors  $u, v$  in  $\mathbb{R}^4$ , where

$$(u \wedge^g v)(w) \doteq g(v, w)u - g(u, w)v \quad (1)$$

---

\*jhanson@digipen.edu

when applied to the four-vector  $w$ . In index notation,  $(u \wedge^g v)_\beta^\alpha = u^\alpha v_\beta - v^\alpha u_\beta$ . Simple bivectors are also called *decomposable bivectors*, and are characterized by the condition  $\det(u \wedge^g v) = 0$ .

While not every Lorentz bivector  $L$  is simple, it is the sum of simple bivectors. In fact, it can be shown that any nonsimple Lorentz bivector  $L$  admits an **orthogonal decomposition**:  $L = L_+ + L_-$ , with  $L_\pm$  simple and  $L_+ L_- = 0 = L_- L_+$ . This decomposition is unique. Indeed, the summands of the decomposition of  $L$  are given by

$$L_\pm = \pm \frac{L^3 - \mu_\mp L}{\mu_+ - \mu_-} \quad (2)$$

where  $\mu_\pm$  are the positive and negative roots of the equation  $x^2 + (\text{tr}_2 L)x + \det L = 0$ , or equivalently, the solutions of the simultaneous equations

$$\mu_+ + \mu_- = -\text{tr}_2 L \quad \text{and} \quad \mu_+ \mu_- = \det L. \quad (3)$$

Here  $\text{tr}_2 L$  is the second order trace of  $L$ , which can be computed by the formula  $\text{tr}_2 L = -\frac{1}{2} \text{tr} L^2$  (this identity holds for any traceless matrix). In particular,  $\text{tr}_2 L_\pm = -\mu_\pm$ . See [3] for details.

## 1.2 Spin representation of the Lorentz algebra

Representations of  $so(g)$  may be constructed from representations of the Clifford algebra  $\mathcal{Cl}(g)$  on  $\mathbb{R}^4$ . Recall that  $\mathcal{Cl}(g)$  is the quotient of the tensor algebra  $T^*(\mathbb{R}^4)$  by the subalgebra generated by the relation  $uv + vu = 2g(u, v)$  for  $u, v \in \mathbb{R}^4$ . Let  $\rho$  be a Clifford algebra representation; i.e., a (possibly complex) vector space  $V$  and a linear map  $\rho : \mathcal{Cl}(g) \rightarrow \text{Hom}(V, V)$  that respects Clifford multiplication:  $\rho(uv) = \rho(u)\rho(v)$ . We obtain the *spin representation*  $\sigma : so(g) \rightarrow \text{Hom}(V, V)$  by setting

$$\sigma(u \wedge^g v) \doteq \frac{1}{4} \rho(uv - vu) \quad (4)$$

for simple Lorentz bivectors, and extending linearly to all of  $so(g)$ . One shows that  $\sigma$  is a Lie algebra homomorphism:  $\sigma([L_1, L_2]) = \sigma(L_1)\sigma(L_2) - \sigma(L_2)\sigma(L_1)$  for all Lorentz bivectors  $L_1, L_2$  (see [2], for example).

A natural choice for the representation  $\rho$  would be gamma matrices; i.e.,  $\rho(u) \doteq u^\alpha \gamma_\alpha$ . However, a representation may be constructed directly from the Clifford algebra itself: view  $V = \mathcal{Cl}(g)$  as a sixteen-dimensional real vector

space, and take  $\rho$  to be the identity. Unlike the gamma matrix representation, this is not an irreducible Clifford algebra representation. In the following, we will not have the need to make a specific choice for  $\rho$ , and we simply refer to  $\sigma$  as “the” spin representation of  $so(g)$ . We remark that all formulas, with the exception of those that appear in section 1.4, are actually valid for summands of the spin representation. In particular, they are valid for half-spin representations.

The key property of the spin representation  $\sigma$  that we will make use of is the following, which makes apparent the usefulness of decomposing a bivector into a sum of simple bivectors. Here we write  $I = \rho(1)$ .

**Theorem 1.** *Suppose  $L = u \wedge^g v$  is a simple Lorentz bivector. Then  $\text{tr}_2 L = g(u, u)g(v, v) - g(u, v)^2$  and  $\sigma(L)^2 = -\frac{1}{4}(\text{tr}_2 L)I$ .*

*Proof.* From equation (1), one computes that  $\text{tr}_2 L = -\frac{1}{2}\text{tr} L^2$  is given by the stated expression (see also [3]). Now compute using equation (4) and the Clifford algebra relation:

$$\begin{aligned}\sigma(u \wedge^g v)^2 &= \frac{1}{16} \rho(uvuv - uv^2u - vu^2v + vuuv) \\ &= \frac{1}{16} \rho(u[-uv + 2g(v, u)]v - g(v, v)g(u, u) \\ &\quad - g(u, u)g(v, v) + v[-vu + 2g(u, v)]u) \\ &= \frac{1}{16} \rho(2g(u, v)(uv + vu) - 4g(u, u)g(v, v))\end{aligned}$$

which implies the stated expression for  $\sigma(L)^2$ . □

### 1.3 Decomposition of a spin representation

If  $L$  is a Lorentz bivector, then  $L^3$  is as well. So we may apply the spin representation directly to each summand in equation (2) to obtain  $\sigma(L_\pm)$  in terms of  $\sigma(L)$  and  $\sigma(L^3)$ . However, we would like an expression that involves only powers of  $\sigma(L)$ .

**Theorem 2.** *If  $L = L_+ + L_-$  is the orthogonal decomposition of a nonsimple Lorentz bivector, then*

$$\sigma(L_\pm) = \frac{\pm 2}{\mu_+ - \mu_-} \left\{ \frac{1}{4}(\mu_\mp + 3\mu_\pm)\sigma(L) - \sigma(L)^3 \right\}$$

with  $\mu_\pm$  as in equation (3).

*Proof.* Since  $(*) \sigma(L) = \sigma(L_+) + \sigma(L_-)$ , we have that  $\sigma(L)^3 = \sigma(L_+)^3 + 3\sigma(L_+)^2\sigma(L_-) + 3\sigma(L_+)\sigma(L_-)^2 + \sigma(L_-)^3$ . Using theorem 1 to reduce powers, we may rewrite this as  $(**) \sigma(L)^3 = \frac{1}{4}(\mu_+ + 3\mu_-)\sigma(L_+) + \frac{1}{4}(\mu_- + 3\mu_+)\sigma(L_-)$ . The determinant of the linear system  $(*)$  and  $(**)$  is  $\frac{1}{2}(\mu_+ - \mu_-)$ , which is nonzero if  $L$  is nonsimple, and the system may be solved to yield the stated expressions for  $\sigma(L_\pm)$ .  $\square$

The summands of the orthogonal decomposition of a nonsimple Lorentz bivector are mutually annihilating. Although their images under the spin representation do not share this property, they do commute.

**Theorem 3.** *If  $L = L_+ + L_-$  is the orthogonal decomposition of the nonsimple Lorentz bivector  $L$ , then  $\sigma(L_+)\sigma(L_-) = \sigma(L_-)\sigma(L_+) = \frac{1}{8}(\text{tr}_2 L)I + \frac{1}{2}\sigma(L)^2$ .*

*Proof.* As  $L_+, L_-$  trivially commute,  $[\sigma(L_+), \sigma(L_-)] = \sigma([L_+, L_-]) = 0$ . By theorem 1 and equation (3), we then have  $\sigma(L)^2 = \sigma(L_+)^2 + 2\sigma(L_+)\sigma(L_-) + \sigma(L_-)^2 = 2\sigma(L_+)\sigma(L_-) - \frac{1}{4}(\text{tr}_2 L)I$ .  $\square$

## 1.4 A computational digression

To use the formula in theorem 2, we need to know the values of  $\mu_\pm$ , which are obtained from the invariants  $\text{tr}_2 L$  and  $\det L$  of  $L$ . However, we would like to deduce these values in the event we only have knowledge of  $\sigma(L)$ .

**Lemma 1.** *For any four-vectors  $a, b, u, v$*

$$\text{tr}\rho(abuv) = \text{tr}I \{g(a, b)g(u, v) - g(a, u)g(b, v) + g(a, v)g(b, u)\}. \quad \square$$

This generalizes the well-known identity for gamma matrices, so we need not repeat the computation here. We do note however, that  $\text{tr}I = \text{tr}\rho(1)$  is the dimension of the representation:  $\text{tr}I = 16$  for the Clifford algebra  $\mathcal{Cl}(g)$  itself, and  $\text{tr}I = 4$  for the gamma matrix representation.

**Lemma 2.** *If  $L = L_+ + L_-$  is the orthogonal decomposition of a Lorentz bivector with  $L_+ = a \wedge^g b$  and  $L_- = u \wedge^g v$ , then*

$$g(a, v)g(b, u) - g(a, u)g(b, v) = 0.$$

*Proof.* From equation (1), one computes  $(a \wedge^g b)(u \wedge^g v) = g(b, u)av^Tg - g(b, v)au^Tg - g(a, u)bv^Tg + g(a, v)bu^Tg$ . Taking the trace of both sides, we get  $\text{tr}\{(a \wedge^g b)(u \wedge^g v)\} = 2g(b, u)g(v, a) - 2g(a, u)g(v, b)$ , which is necessarily zero, since  $(a \wedge^g b)(u \wedge^g v) = L_+L_- = 0$ .  $\square$

**Theorem 4.** *If  $L = L_+ + L_-$  is the orthogonal decomposition of a Lorentz bivector, then  $\text{tr}\{\sigma(L_+)\sigma(L_-)\} = 0$ .*

*Proof.* Write  $L_+ = a \wedge^g b$  and  $L_- = u \wedge^g v$ . Then  $\sigma(L_+)\sigma(L_-) = \frac{1}{16}\rho(ab - ba)\rho(uv - vu) = \frac{1}{16}(\rho(abuv) - \rho(abvu) - \rho(bauv) + \rho(bavu))$ . Take the trace and apply the previous two lemmas.  $\square$

**Theorem 5.** *For any Lorentz bivector  $L$ ,  $\text{tr}_2 L = -4 \text{tr}\sigma(L)^2 / \text{tr} I$  and  $\det L = 4 \text{tr}\sigma(L)^4 / \text{tr} I - 4 \text{tr}^2 \sigma(L)^2 / \text{tr}^2 I$ .*

*Proof.* For the first formula, take the trace of the formula in theorem 3. For the second, use the orthogonal decomposition  $L = L_+ + L_-$  and theorem 1 to compute  $\sigma(L)^4 = \{\sigma(L_+) + \sigma(L_-)\}^4 = \frac{1}{16}\mu_+^2 I - \mu_+ \sigma(L_+) \sigma(L_-) + \frac{3}{8}\mu_+ \mu_- I - \mu_- \sigma(L_+) \sigma(L_-) + \frac{1}{16}\mu_-^2 I$ . Taking traces, we get  $\text{tr}\sigma(L)^4 = \frac{1}{16}(\text{tr} I)(\mu_+^2 + 6\mu_+ \mu_- + \mu_-^2) = \frac{1}{16}(\text{tr} I)\{(\mu_+ + \mu_-)^2 + 4\mu_+ \mu_-\} = \frac{1}{16}(\text{tr} I)(\text{tr}_2^2 L + 4 \det L)$ , courtesy of equation (3). Solving for  $\det L$  and using the first formula yields the desired expression.  $\square$

## 2 Exponential of the spin representation

By general principles, the exponential operation  $\exp(L) \doteq \sum_{n \geq 0} L^n / n!$  is a map  $\exp : so(g) \rightarrow O(g)$ , whose image is the connected component  $SO^+(g)$  of  $O(g)$  containing the identity transformation; i.e., the set of all proper orthochronous Lorentz transformations. A closed formula for  $\exp(L)$  was obtained in [1]. Here we give a closed expression for  $\exp(\sigma(L))$  for both simple and nonsimple Lorentz bivectors.

**Theorem 6.** *If  $L$  is a simple Lorentz bivector, then  $\exp(\sigma(L)) = \bar{c} + \bar{s}\sigma(L)$ , where*

$$\begin{aligned} \text{if } \text{tr}_2 L > 0, \quad \bar{c} &\doteq \cos \frac{1}{2} \sqrt{\text{tr}_2 L} & \text{and} \quad \bar{s} &\doteq \frac{2}{\sqrt{\text{tr}_2 L}} \sin \frac{1}{2} \sqrt{\text{tr}_2 L} \\ \text{if } \text{tr}_2 L < 0, \quad \bar{c} &\doteq \cosh \frac{1}{2} \sqrt{-\text{tr}_2 L} & \text{and} \quad \bar{s} &\doteq \frac{2}{\sqrt{-\text{tr}_2 L}} \sinh \frac{1}{2} \sqrt{-\text{tr}_2 L} \end{aligned}$$

and if  $\text{tr}_2 L = 0$ , then  $\bar{c} = 1 = \bar{s}$ .

*Proof.* In the case  $\text{tr}_2 L > 0$ , theorem 1 implies  $\sigma(L)^{2p} = (-\theta^2)^p = (-1)^p \theta^{2p}$ , where  $\theta \doteq \frac{1}{2} \sqrt{\text{tr}_2 L}$ . Consequently, we may write  $\sigma(L)^{2p+1} = \sigma(L)^{2p} \sigma(L) = (-1)^p \theta^{2p+1} \sigma(L) / \theta$ . Thus the series  $\sum_{n \geq 0} \sigma(L)^n / n! = \sum_{p \geq 0} \sigma(L)^{2p} / (2p)! +$

$\sum_{p \geq 0} \sigma(L)^{2p+1}/(2p+1)!$  is summed using the usual Taylor series expansion for sine and cosine. The case when  $\text{tr}_2 L < 0$  is similar, and the case  $\text{tr}_2 L = 0$  is trivial.  $\square$

Recall that  $\exp(A+B) = \exp(A)\exp(B)$  whenever the matrices  $A, B$  commute. Thus,  $\exp(\sigma(L)) = \exp(\sigma(L_+))\exp(\sigma(L_-))$ . Theorems 3 and 6 then lead to the following.

**Theorem 7.** *Suppose  $L = L_+ + L_-$  is the orthogonal decomposition of a nonsimple Lorentz bivector. Define  $\theta_{\pm} \doteq \frac{1}{2}\sqrt{\mp \text{tr}_2 L_{\pm}}$ ,  $\bar{c}_+ \doteq \cosh \theta_+$ ,  $\bar{c}_- \doteq \cos \theta_-$ ,  $\bar{s}_+ \doteq \sinh \theta_+/\theta_+$ , and  $\bar{s}_- \doteq \sin \theta_-/\theta_-$ . Then*

$$\exp(\sigma(L)) = \bar{c}_+\bar{c}_- + \bar{s}_+\bar{c}_- \sigma(L_+) + \bar{c}_+\bar{s}_- \sigma(L_-) + \bar{s}_+\bar{s}_- \sigma(L_+)\sigma(L_-) \quad \square$$

We derive an alternative formula for  $\exp(\sigma(L))$  as a polynomial in  $\sigma(L)$ . By combining theorems 2 and 3 with theorem 7, we obtain the following.

**Theorem 8.** *If  $L$  is a nonsimple Lorentz bivector, then*

$$\exp(\sigma(L)) = \alpha_0 + \alpha_1 \sigma(L) + \alpha_2 \sigma(L)^2 + \alpha_3 \sigma(L)^3$$

$$\begin{aligned} \alpha_0 &\doteq \bar{c}_+\bar{c}_- - \frac{1}{8}(\mu_+ + \mu_-)\bar{s}_+\bar{s}_- & \alpha_2 &\doteq \frac{1}{2}\bar{s}_+\bar{s}_- \\ \alpha_1 &\doteq \frac{1}{4}N\{(\mu_- + 3\mu_+)\bar{s}_+\bar{c}_- - (\mu_+ + 3\mu_-)\bar{c}_+\bar{s}_-\} & \alpha_3 &\doteq N(\bar{c}_+\bar{s}_- - \bar{s}_+\bar{c}_-) \end{aligned}$$

with  $\mu_{\pm}$  as in equation (3),  $\bar{c}_{\pm}, \bar{s}_{\pm}$  as in theorem 7, and  $N \doteq 2/(\mu_+ - \mu_-)$ .  $\square$

### 3 Spin representation of a Lorentz transformation

The spin representation  $\sigma : so(g) \rightarrow \text{Hom}(V, V)$  on the Lie algebra level induces a projective representation  $\Sigma : SO^+(g) \rightarrow \text{SL}(V)/\pm$  on the Lie group level ([4]). We would like to deduce an explicit formula for  $\Sigma$ .

We define a Lorentz transformation  $\Lambda \in SO^+(g)$  to be **simple** if it is the image of a simple Lorentz bivector under the exponential map. A criterion for simplicity is that  $\text{tr}_2 \Lambda = 2(\text{tr} \Lambda - 1)$ . Here, the second order trace may be computed from the general formula  $\text{tr}_2 \Lambda = \frac{1}{2}(\text{tr}^2 \Lambda - \text{tr} \Lambda^2)$ . Moreover, it should be noted that for any proper orthochronous Lorentz transformation (simple or not),  $\text{tr} \Lambda \geq 0$ . See [3] for more details.

We will need the following fact for computing the logarithm of a simple Lorentz transformation, as given in [3]. The special case when  $\text{tr}\Lambda = 0$  will be handled later.

**Proposition 1.** *If  $\Lambda \in SO^+(g)$  is a simple Lorentz transformation with  $\text{tr}\Lambda > 0$ , then  $\Lambda = \exp(L)$  and  $\text{tr}_2 L = -\mu$ , where  $L = \frac{1}{2}k(\Lambda - \Lambda^{-1})$  and*

1. *if  $0 < \text{tr}\Lambda < 4$ , then  $k = \frac{\sqrt{-\mu}}{\sin \sqrt{-\mu}}$  and  $\sqrt{-\mu} = \cos^{-1}(\frac{1}{2}\text{tr}\Lambda - 1)$ ,*
2. *if  $\text{tr}\Lambda > 4$ , then  $k = \frac{\sqrt{\mu}}{\sinh \sqrt{\mu}}$  and  $\sqrt{\mu} = \cosh^{-1}(\frac{1}{2}\text{tr}\Lambda - 1)$ ,*
3. *if  $\text{tr}\Lambda = 4$ , then  $k = 0$  and  $\mu = 0$ .*

**Theorem 9.** *Suppose  $\Lambda \in SO^+(g)$  is simple. If  $\text{tr}\Lambda > 0$ , then up to an overall sign,*

$$\Sigma(\Lambda) = \frac{1}{2\sqrt{\text{tr}\Lambda}} \left\{ \text{tr}\Lambda + 2\sigma(\Lambda - \Lambda^{-1}) \right\}.$$

*Proof.* Let  $L$  be a simple Lorentz bivector such that  $\Lambda = \exp(L)$ . By the general properties of the exponential map on a Lie algebra,  $\Sigma(\Lambda) = \exp(\sigma(L))$ . Writing  $L = \frac{1}{2}k(\Lambda - \Lambda^{-1})$  as in proposition 1, we have  $\exp(\sigma(L)) = \bar{c} + \frac{1}{2}k\bar{s}\sigma(\Lambda - \Lambda^{-1})$ , according to theorem 6. The values of  $\bar{c}$ ,  $\bar{s}$  depend on the value of  $\text{tr}_2 L$ . We consider the case  $\text{tr}_2 L > 0$ , so that  $\mu < 0$  in the notation of proposition 1, which occurs when  $0 < \text{tr}\Lambda < 4$ . The other two cases are similar. In this case, we have  $\cos \sqrt{-\mu} = \frac{1}{2}\text{tr}\Lambda - 1$ . Now,  $\bar{c} = \cos \frac{1}{2}\sqrt{\text{tr}_2 L} = \cos \frac{1}{2}\sqrt{-\mu}$ . Similarly,

$$\frac{1}{2}k\bar{s} = \frac{1}{2} \frac{\sqrt{-\mu}}{\sin \sqrt{-\mu}} \frac{2}{\sqrt{-\mu}} \sin \frac{1}{2}\sqrt{-\mu} = \frac{\sin \frac{1}{2}\sqrt{-\mu}}{\sin \sqrt{-\mu}} = \frac{1}{2 \cos \frac{1}{2}\sqrt{-\mu}}$$

On the other hand, we have  $\cos^2 \frac{1}{2}\sqrt{-\mu} = \frac{1}{2}(1 - \cos \sqrt{-\mu}) = \frac{1}{4}\text{tr}\Lambda$ , so that  $\cos \frac{1}{2}\sqrt{-\mu} = \pm \frac{1}{2}\sqrt{\text{tr}\Lambda}$ . Although the choice of  $L$  determines the sign here, the choice of  $L$  such that  $\exp(L) = \Lambda$  is not unique. Indeed, if we take  $L' = \alpha L$ , with  $\alpha$  chosen such that  $\sqrt{\text{tr}_2 L'} = \sqrt{\text{tr}_2 L} + 2\pi$ , then  $\exp(L') = \Lambda$ . However,  $\frac{1}{2}\sqrt{-\mu'} = \frac{1}{2}\sqrt{-\mu} + \pi$ , so that  $\exp(\sigma(L')) = -\exp(\sigma(L))$ .  $\square$

To obtain an analogous formula for a nonsimple Lorentz transformation, we will make use of the fact that such a transformation is a product of

commuting simple transformations. Indeed, since  $SO^+(g)$  is exponential, we may write  $\Lambda = \exp(L)$  for some Lorentz bivector  $L$ . Using the orthogonal decomposition  $L = L_+ + L_-$ , we have  $\exp(L) = \exp(L_+) \exp(L_-)$ . Taking  $\Lambda_{\pm} \doteq \exp(L_{\pm})$ , we may write  $\Lambda = \Lambda_+ \Lambda_-$ , with  $\Lambda_+, \Lambda_-$  commuting simple Lorentz transformations. In [3], the following explicit formula is obtained.

**Proposition 2.** *If  $\Lambda \in SO^+(g)$  is nonsimple, then  $\Lambda = \Lambda_+ \Lambda_-$ , where  $\Lambda_{\pm}$  are the commuting simple Lorentz transformations*

$$\Lambda_{\pm} = \pm \frac{1}{2(c_+ - c_-)} \{ (1 + 2c_{\pm})I - \Lambda^{-1} - (1 + 2c_{\mp})\Lambda + \Lambda^2 \}$$

with  $c_{\pm} = \frac{1}{4}(\text{tr}\Lambda \pm \sqrt{\Delta})$  and  $\Delta \doteq \text{tr}^2\Lambda - 4\text{tr}_2\Lambda + 8$ ,  $c_+ > 1$ , and  $1 \leq c_- < 1$ .

Using this decomposition, we use the fact that  $\Sigma$  is a group homomorphism to write  $\Sigma(\Lambda) = \Sigma(\Lambda_+)\Sigma(\Lambda_-)$ , where each factor on the right hand side may be computed using theorem 9. Note that  $\Sigma(\Lambda_{\pm})$  necessarily commute.

**Theorem 10.** *If  $\Lambda$  is a nonsimple proper orthochronous Lorentz transformation with  $2 + 2\text{tr}\Lambda + \text{tr}_2\Lambda \neq 0$ , then up to sign*

$$\begin{aligned} \Sigma(\Lambda) = \frac{1}{2\sqrt{2 + 2\text{tr}\Lambda + \text{tr}_2\Lambda}} \{ & (2 + \text{tr}\Lambda + \text{tr}_2\Lambda - \frac{1}{4}\text{tr}^2\Lambda) \\ & + (\text{tr}\Lambda + 2)\sigma(\Lambda - \Lambda^{-1}) - \sigma(\Lambda^2 - \Lambda^{-2}) + \sigma(\Lambda - \Lambda^{-1})^2 \} \end{aligned}$$

*Proof.* For brevity, we set  $\tau_k^{\pm} \doteq \text{tr}_k\Lambda_{\pm}$ . From theorem 9, we compute  $\Sigma(\Lambda) = \Sigma(\Lambda_+)\Sigma(\Lambda_-)$  to be:

$$\begin{aligned} \Sigma(\Lambda) = \frac{1}{4\sqrt{\tau_1^+\tau_1^-}} \{ & \tau_1^+\tau_1^- + 2\tau_1^+\sigma(\Lambda_- - \Lambda_-^{-1}) + 2\tau_1^-\sigma(\Lambda_+ - \Lambda_+^{-1}) \\ & + 4\sigma(\Lambda_+ - \Lambda_+^{-1})\sigma(\Lambda_- - \Lambda_-^{-1}) \} \quad (5) \end{aligned}$$

Now from proposition 2, one shows that  $(\star)$   $\tau_1 \doteq \text{tr}\Lambda = 2(c_+ + c_-)$  and  $\tau_2 \doteq \text{tr}_2\Lambda = 4c_+c_- + 2$ , and that  $(\star\star)$   $\tau_1^{\pm} = 2(1 + c_{\pm})$ . Moreover, since for any Lorentz transformation  $\Lambda^{-1} = g^{-1}\Lambda^Tg$ , we find that

$$\Lambda_{\pm} - \Lambda_{\pm}^{-1} = \mp \frac{1}{2(c_+ - c_-)} \{ 2c_{\mp}(\Lambda - \Lambda^{-1}) - (\Lambda^2 - \Lambda^{-2}) \} \quad (6)$$

We now rewrite the summands of equation (5) in terms of  $\Lambda$  and its first and second order traces. For the first summand, using  $(\star\star)$  and  $(\star)$  we obtain

$$\tau_1^+\tau_1^- = 4(1 + c_+)(1 + c_-) = 2 + 2\tau_1 + \tau_2 \quad (7)$$



For the second and third summands in (5), using  $(\star\star)$  and (6) one computes  $\tau_1^+ \sigma(\Lambda_- - \Lambda_-^{-1}) + \tau_1^- \sigma(\Lambda_+ - \Lambda_+^{-1}) = 2(1 + c_+ + c_-) \sigma(\Lambda - \Lambda^{-1}) - \sigma(\Lambda^2 - \Lambda^{-2})$ . Thus by  $(\star)$ ,

$$\tau_1^+ \sigma(\Lambda_- - \Lambda_-^{-1}) + \tau_1^- \sigma(\Lambda_+ - \Lambda_+^{-1}) = (\tau_1 + 2) \sigma(\Lambda - \Lambda^{-1}) - \sigma(\Lambda^2 - \Lambda^{-2}) \quad (8)$$

For the fourth summand in (5), we similarly compute (although we also need to use the fact that  $\sigma(\Lambda - \Lambda^{-1})$  and  $\sigma(\Lambda^2 - \Lambda^{-2})$  commute, as  $\sigma$  is a Lie algebra homomorphism)

$$4\sigma(\Lambda_+ - \Lambda_+^{-1})\sigma(\Lambda_- - \Lambda_-^{-1}) = \frac{1}{(c_+ - c_-)^2} \left\{ (2 - \tau_2) \sigma(\Lambda - \Lambda^{-1})^2 + \tau_1 \sigma(\Lambda - \Lambda^{-1}) \sigma(\Lambda^2 - \Lambda^{-2}) - \sigma(\Lambda^2 - \Lambda^{-2})^2 \right\} \quad (9)$$

However, the terms in this expression are not algebraically independent. To find a relation, we make use of the fact that  $\Lambda_{\pm}$  are simple: from theorem 1,  $\sigma(L_{\pm})^2 = -\frac{1}{4} \text{tr}_2 L_{\pm}$ , where  $L_{\pm}$  are such that  $\exp(L_{\pm}) = \Lambda_{\pm}$ , and may be computed using proposition 1. Indeed,  $L_+ = \frac{1}{2} k_+ (\Lambda_+ - \Lambda_+^{-1})$ , with  $\text{tr}_2 L_+ = -\mu_+$ ,  $c_+ = \cosh \sqrt{\mu_+}$ , and  $k_+ = \sqrt{\mu_+} / \sinh \sqrt{\mu_+}$ . The equation  $\sigma(L_+)^2 = -\frac{1}{4} \text{tr}_2 L_+$  and (6) then lead to

$$4c_-^2 \sigma(\Lambda - \Lambda^{-1})^2 - 4c_- \sigma(\Lambda - \Lambda^{-1}) \sigma(\Lambda^2 - \Lambda^{-2}) + \sigma(\Lambda^2 - \Lambda^{-2})^2 = 4(c_+ - c_-)^2 (c_+^2 - 1)$$

(note that  $\sinh^2 \sqrt{\mu_+} = c_+^2 - 1$ ). Similarly,  $\sigma(L_-)^2 = -\frac{1}{4} \text{tr}_2 L_-$ , proposition 1, and (6) imply

$$4c_+^2 \sigma(\Lambda - \Lambda^{-1})^2 - 4c_+ \sigma(\Lambda - \Lambda^{-1}) \sigma(\Lambda^2 - \Lambda^{-2}) + \sigma(\Lambda^2 - \Lambda^{-2})^2 = 4(c_+ - c_-)^2 (c_-^2 - 1)$$

Adding these two equations together and using  $(\star)$  then yields the relation

$$(\tau_1^2 - 2\tau_2 + 4) \sigma(\Lambda - \Lambda^{-1})^2 - 2\tau_1 \sigma(\Lambda - \Lambda^{-1}) \sigma(\Lambda^2 - \Lambda^{-2}) + 2\sigma(\Lambda^2 - \Lambda^{-2})^2 = (c_+ - c_-)^2 (\tau_1^2 - 2\tau_2 - 4) \quad (10)$$

Combining (9) and (10) together, we see that we may write the fourth summand in (5) as

$$4\sigma(\Lambda_+ - \Lambda_+^{-1})\sigma(\Lambda_- - \Lambda_-^{-1}) = 2\sigma(\Lambda - \Lambda^{-1})^2 - \frac{1}{2}(\tau_1^2 - 2\tau_2 - 4) \quad (11)$$

Combining (5), (7), (8), and (11) give the desired formula for  $\Sigma(\Lambda)$ .  $\square$

### 3.1 Special case

We consider the case when a Lorentz transformation  $\Lambda \in SO^+(g)$  satisfies the identity

$$2 + 2\text{tr}\Lambda + \text{tr}_2\Lambda = 0 \quad (12)$$

In the case when  $\Lambda$  is simple, so that  $\text{tr}_2\Lambda = 2(\text{tr}\Lambda - 1)$ , this condition reduces to  $\text{tr}\Lambda = 0$ . It should be noted that a simple Lorentz transformation is traceless only if  $\Lambda = \exp(L)$  for some simple Lorentz bivector with  $\text{tr}_2L = \pi^2$ . A nonsimple transformation satisfies (12) only if its decomposition in proposition 2,  $\Lambda = \Lambda_+\Lambda_-$ , is such that the simple factor  $\Lambda_-$  is traceless.

We will not be able to obtain an explicit formula for the spin representation  $\Sigma(\Lambda)$  of such a Lorentz transformation. However, we can give an algorithm. The key lies in the following two facts from [3].

**Proposition 3.** *Let  $u, v$  be four-vectors, and set  $L \doteq u \wedge^g v$ . The two-plane  $\mathcal{P}$  spanned by  $u, v$  is nondegenerate (that is,  $g$  is nondegenerate when restricted to  $\mathcal{P}$ ) if and only if  $\text{tr}_2L \neq 0$ , in which case  $P_L \doteq -L^2/\text{tr}_2L$  is  $g$ -orthogonal projection onto  $\mathcal{P}$ . Conversely, if  $P$  is  $g$ -orthogonal projection onto a two-plane, then the two-plane is nondegenerate and  $P = P_L$ , where  $L = u \wedge^g v$  and  $u, v$  are any linearly independent four-vectors in the image of  $P$ .*

**Proposition 4.** *If  $\Lambda$  is a simple Lorentz transformation with  $\text{tr}\Lambda = 0$ , then  $\Lambda^2 = I$ ,  $P_\Lambda \doteq \frac{1}{2}(I - \Lambda)$  is  $g$ -orthogonal projection onto a nondegenerate two-plane, and  $-\pi^2 P_\Lambda$  is the square of a simple Lorentz bivector  $L_\Lambda$  with  $\exp(L_\Lambda) = \Lambda$  and  $\text{tr}_2L_\Lambda = \pi^2$ .*

**Theorem 11.** *Suppose  $\Lambda \in SO^+(g)$  is simple with  $\text{tr}\Lambda = 0$ . Let  $u, v$  be any two linearly independent vectors in the image of  $P_\Lambda = \frac{1}{2}(I - \Lambda)$ . Then up to sign,  $\Sigma(\Lambda) = (2/\sqrt{\text{tr}_2(u \wedge^g v)})\sigma(u \wedge^g v)$ .*

*Proof.* Set  $L \doteq u \wedge^g v$ . Then  $P_L$  is  $g$ -orthogonal projection onto the (necessarily nondegenerate) two-plane  $\mathcal{P}$  spanned by  $u, v$ . By uniqueness of  $g$ -orthogonal projection, we must have  $P_L = P_\Lambda$ . Now  $L_\Lambda = u' \wedge^g v'$ , where  $u', v'$  lie in  $\mathcal{P}$ . Writing  $u' = au + bv$  and  $v' = cu + dv$ , we compute that  $u' \wedge^g v' = (ad - bc)u \wedge^g v$ ; i.e.,  $L_\Lambda = \alpha L$  for some scalar  $\alpha$ . Taking 2-traces, we get  $\pi^2 = \text{tr}_2L_\Lambda = \alpha^2\text{tr}_2L$ , so that  $\alpha = \pm\pi/\sqrt{\text{tr}_2L}$ . Since  $\exp(L_\Lambda) = \Lambda$ ,  $\Sigma(\Lambda) = \exp\sigma(L_\Lambda)$ . On the other hand, by theorem 6,  $\exp\sigma(L_\Lambda) = (2/\pi)\sigma(L_\Lambda) = (2\alpha/\pi)\sigma(L)$ .  $\square$

## References

- [1] Bortolomé Coll and Fernando San José, *On the exponential of the 2-forms in relativity*, General Relativity and Gravitation, 22 (7), 811–826, 1990.
- [2] William Fulton and Joe Harris, *Representation Theory*, Springer–Verlag, 1991.
- [3] jason hanson, *Orthogonal decomposition of Lorentz transformations*, arXiv:1103.1072v1 [gr-qc].
- [4] Eugene Wigner, *Unitary representations of the inhomogeneous Lorentz group*, Annals of Mathematics, 40, 149–204, 1939.